Communications

Difficulties in basic arithmetic and geometry as related to school algebra and the current effect of 'dematheatization'

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We argue that educational practice in mathematics is not as long lasting as is sometimes claimed. It is intimately related to *prevalent social representations* of what is 'useful knowledge', that is, what decision-makers and teachers consider to be knowledge and skills necessary for all students. Basic arithmetic and geometry, which until the sixties were necessary in commercial life, as practical methods for solving problems, drawing figures and computing, cease to be indispensable in education from the moment that calculators and computers 'perform' arithmetic operations and 'draw' geometrical figures.

In recent decades, the educational direction of the leading technocratic elite in USA and the European Union turned toward hybrid knowledge and skills in order to train future citizens to adapt to technological developments and socio-economic changes. For example, students should be able to recognize structured data in a pictorial form (*e.g.,* circular diagrams or graphs). At the same time, there is an international shift towards 'practical' mathematical problems related to the workplace [1] and educational activities that incorporate preexisting technological tools in teaching [2]. Both of these turns seem to be aspects of a single, widespread change of direction in international educational policy.

Part of this change is an increased emphasis on modeling and simulation of 'real' activities. As Harouni (2015, p. 62) points out, the actual commercial-administrative content of school mathematics was not made visible to teachers and students until recently. The shift to 'real' problems makes the role of mathematics in an uncertain capitalist world more clear. We consider this instrumental adaptation of mathematical knowledge, in a procedural/algorithmic form suitable for a virtual introduction of students to labor or marketing activities, to be *dematheatization* (Keitel, 1989; Gellert & Jablonka, 2009), a new *technological* form of commercial-administrative type of mathematics education.

The aim of this communication is to discuss these changes in parallel with the practice of school algebra and to present some cognitive consequences, revealed by a test given to Greek students.

The dematheatization effect of informational technology and school algebra

The international shift in the design of new teaching activities, which incorporates calculators and pre-existing educational software, means that geometrical constructions as well as representation of three-dimensional space are no longer left to human perception and performance, shifting instead to computer processing.

The phenomenon of dematheatization by technology (Gellert & Jablonka, 2009) is a replacement of thinking and mathematizing processes by technological 'black boxes', *i.e.,* programs that simulate mathematical constructions and thus do not requirethier users to understand their underlying mathematical structures. Christine Keitel, as early as in 1989, had already observed that, "the hand-calculator is the culmination so far of a development by which, while reality is being structured more and more by mathematics, the average individual is more and more relieved of the need to use mathematics" (p. 9). She adds, "Dematheatisation is brought about by the very existence of the products of our technologically-structured environment: dematheatisation is inherent in these products as it is in technology" (p. 9).

In parallel to these developments, school algebra—in contrast to basic arithmetic—offers a kind of instrumental adaptation of mathematical knowledge similar to that of informational technology. For example, this is evident in the following word problem, extracted from the official textbook of the third year of the Greek Junior High School:

> Trousers cost from 30€ to 35€ and a T-shirt costs from 20€ to 25€. If someone wants to buy 2 trousers and 3 T-shirts, determine the maximum and minimum of the amount he has to pay.

The solution expected goes in the following lines: Let x be the price of trousers and y the price of T-shirts. Then $30 < x < 35$ and $20 < y < 25$. Then for the price of two trousers we have $60 < 2x < 70$ and for the price of three T-shirts $60 < 3y < 75$. Adding the two inequalities we obtain $120 < 2x+3y < 145$, so we have to pay between 120€ to 145€. Blindly following memorized rules of algebra to solve such a problem is another kind of dematheatization.

Technology need not dematheatize. Under special pedagogical conditions, some mathematics educational software could contribute to heuristics, exploration and visualization as well as to mathematical experimentation (Hanna, 2000) and testing conjectures, thus revealing a creative aspect of information technology. Especially *Logo* (Papert, 1980) and dynamic geometry environments could encourage students to 'play' with data and the questions asked, to explore and experience mathematics. Thus, the traditional sequence of algebra textbooks (axioms & definitions ⟶ theorems ⟶ exercises) could be replaced by searching, conjecturing, formulating, and proving or refuting.

A diagnostic test and its results

Greek students are exposed to instrumental procedures in school algebra as well as to the effect of dematheatization by technology (as described above). Hence, we decided to pose the following problem to first-year mathematics students at the Universities of Patras and Thessaly.

> A box has a shape of rectangular parallelepiped, with integer numbers *m, n, k* as its sides. We want to fill the box, without leaving spaces, with equal cubes using the least possible number. How many cubes will we need?

We gave the same problem to junior and senior high school students in a semi-urban area of Greece in the following form:

> A box has a rectangular parallelepiped shape, with dimensions 6dm, 9dm and 15dm. We want to fill (without leaving spaces) the box with equal cubes, using the least possible number.
>
> i. How long should the edge of each cube be?
>
> ii. How many cubes will fill the box?

These problems were posed as a diagnostic test. Would the students face the problem by combining their arithmetical and geometric knowledge? Would they represent the problem with a geometrical figure? What would be their difficulties?

Most university students thought that the number of cubes required is the least common multiple of the numbers *m, n, k,* perhaps because we asked for 'the least possible number' of cubes in the problem. Many students calculated the LCM of the numbers *m, n, k* as their product. Some students even replied that the least possible number of cubes would be *two*, because we can cut the parallelepiped at the middle in two equal parts.

Many high school students confused volume with area. For example, they drew a rectangle and tried to fill it with squares. A possible reason for this behavior is that geometry in Greek high schools covers only plane figures and not solids.

Dimitris, a senior high school student, showed a detachment from the meaning of the problem. He wrote:

> The volume of the parallelepiped is $V_{par} = 6{\cdot}9{\cdot}15 = 810$. If x stands for the edge of each cube and y for the number of cubes requested, then the volume of the cube will be $V_{cube} = x^3$. Therefore $V_{par}/V_{cube} = 810/x^3 = y$. Also the area [presumably of a particular side] of the parallelepiped is $E_{par} = 15{\cdot}6 = 90$, and the area [of a side] of the cube is $E_{cube} = x^2$, hence $E_{par}/E_{cube} = 90/x^2 = y$.

Solving the system of the two equations he found led Dimitris to the conclusion $x = 9$ and $y = 9 + {}^1/_9$ (a fractional number of cubes?).

In contrast to the responses above, there was one student in the first year of junior high school who answered the question by understanding the algorithms used. He wrote, "Each side would be divided with the greatest common divisor of 6, 9 and 15, that is 3" and he continued, "The result of the previous calculation is the denominator, in every fraction with the numerator being one of the three sides. So we have $({}^6/_3){\cdot}({}^9/_3){\cdot}({}^{15}/_3) = 2{\cdot}3{\cdot}5 = 30$ cubes."

Some concluding remarks and suggestions

According to the test, students found it difficult to work in three dimensions. Either they were unable to make the figure (which would have given them a useful representation) or they completely ignored the third dimension. This reminds us of the 'six-matches problem' (Murray & Byrne, 2013), in which the assumtion that one must solve the problem in two dimensions makes it impossible. These students' reactions are more or less expected since three-dimensional geometrical activities are generally missing from Greek school mathematics.

Some students, in our test, were led to an arbitrary algebraic reformulation of the problem and subsequently to the loss of meaning of mathematical concepts involved. Other results apparent in students' responses are the inability to understand a problem that combines geometry with arithmetic, and their difficulty in connecting the data of the problem with appropriate concepts and procedures. These difficulties stem from two different hypothetical aspects of such problems. On the one hand, they are considered as problems of arithmetic, whereby key roles are reserved for basic operations, such as finding the greatest common divisor and least common multiple. On the other hand, there is a structured system, algebra, for operating on these problems with variables and formal mathematical expressions. This creates confusion among the students due to their inability to understand symbolic procedures and apply them to concrete data.

We neither support rote or formal learning of arithmetic and geometry, nor the replacement of it with 'black box' technology. On the contrary, we believe that information technology can support the learning process by leading students to conjectures and counter-examples, under some conditions. In this direction we suggest that attention has to be paid at the following points:

a) Information technology should not be a substitute for human acts of mental calculation and geometric construction.

b) Information technology should not be a substitute for students' experience in three-dimensional space.

c) Users of technological applications (both learners and teachers) should understand mathematical structures (in our case arithmetic operations and relationships or geometric transformations) and the conditions under which they can be applied.

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Notes

[1] See for example the *MaSciL* project (Mathematics and Science for Life), at https://mascil-project.ph-freiburg.de/.

[2] An example for such an experiment in Greece, as early as in 1990, took place with Dr Panayota Tringa, a former assistant in the Department of Statistics in Athens University of Economics and Business, whose PhD thesis focused on the use of computers in teaching basic arithmetical concepts as the greatest common divisor.

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Productive struggle in mathematics

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The idea of *productive struggle* may seem self-contradictory since the word *struggle* is not often associated with being productive. However, it has been present in mathematics education theory for a long time. It can be found in the foundational writings of Piaget, Vygotsky and Dewey, and is central to romanticist, progressivist, and authentic-learning models of education. Productive struggle has been 'rediscovered' many times and called by many names which can conceal some important connections. In this communication I explore some of those connections and pull together some of the many phrases currently in play that refer to the same phenomenon.

In the field of mathematics education, Hiebert & Grouws (2007) proposed that struggle is a necessary condition for mathematical sense-making while Betts & Rosenberg (2016) go as far as to say that productive struggle is such a fundamental aspect of problem solving, that if a learner does not struggle, problem solving does not take place. Productive struggle, therefore, is akin to what it means to *do* mathematics (Warshauer, 2015), and the result is having more powerful, useful and flexible ways of making sense of mathematics. Often terms such as "controlled floundering" (Pogrow in Sullivan *et al.,* 2015, p. 20) or a "zone of confusion" (Livy, Muir & Sullivan, 2018, p. 21) are also used to describe productive struggle. Pauli (1960) suggests that a common feature in all authentic problem-solving situations is initial confusion that gives rise to understanding as the learner works through the problem. What is not desirable in mathematics classrooms is unproductive success which is the "illusion of learning" when learners follow drilled procedures or rote calculations (Kapur, 2016, p. 290) to achieve 'correct' answers without a real understanding of the concepts involved. A teacher providing excessive control and scaffolding (*i.e.,* telling and showing every step) typifies unproductive success as a notion that is in contrast to productive struggle.

Some features of productive struggle

Sullivan *et al.* (2015) elaborate some of features of productive struggle, such as sustained thinking, decision-making and risk taking. To keep learners in their "zone of productive struggle" (Townsend, Slavitt & McDuffe, 2018, p. 217), four more specific and interrelated elements affecting learning will be discussed: classroom environments, task features, teacher orientations, and learner orientations.

Classroom environments

Kapur and Bielaczyc (2012) propose an 'explore-first-then-instruct' process. They provide three essential design features for mathematics classrooms. Teachers should provide complex problems that require multiple representation and solution methods; allow for opportunities to explain and elaborate; and allow for opportunities to compare and contrast the representations and solution methods, not unlike some elements of variation theory.

Warshauer (2015) suggests environments that are non-judgmental of failed attempts to support learners' productive work. Teachers perpetuate environments that inhibit productive struggle through teacher lust (Tyminski, 2010) which is the desire to tell or show learners exactly what to do, typically through a path-smoothing model (Wigley, 1992) that helps learners avoid time-consuming struggles (Granberg, 2016). The classroom environment should be one in which mistakes are expected and valued for their learning potential.

Tasks

Productive struggle can be fostered through engaging with challenging tasks (Hiebert & Grouws, 2007; Perkins, 2016; Sullivan *et al.,* 2015; Cheeseman, Roche & Walker, 2016). Challenging tasks produce anxiety and extensive cognitive effort (for both learners and teachers), largely due to the "unpredictable nature of the solution process required" (Smith & Stein, 1998, p. 348).

Dewey (1910) understood that teachers have the complex undertaking of balancing tasks so that both easy and difficult thinking are positioned alongside each other. If the task is too easy, there is no need for deep thinking; but, if the task is too difficult, it may result in feelings of hopelessness. While Bruner's (1960) principle that tasks should be *developmentally appropriate* is sound, Kapur (2016, p. 293) suggests the following criteria for tasks to enable productive struggle:

1. The task should challenge the learner to explore, but not give up.
2. It should allow multiple solutions, strategies and representations, i.e. allow space for exploration.
3. It should activate learners' prior knowledge.
4. It should allow the teacher to compare and contrast different solutions to highlight critical features of the concepts at hand.

Therefore, tasks that elicit productive struggle should be carefully designed to be both set in learners' current ways of thinking but also to enable them to extend their thinking through challenge.

Teacher's role during productive struggle

The critical role of the teacher in learning mathematics is well established (Muijs & Reynolds, 2002). Although the teacher is the key to providing challenge and motivation that learners need to learn (Winstanley, 2010), teachers do need specific guidance in providing and sustaining challenge.

Brousseau (1997) calls for an "adidactical situation" (p. 30) in which the teacher should devolve problems to the learner and the learner should accept responsibility for solving them, knowing that they have been selected by the teacher for learning. The teacher should not interfere with the learning process by providing or suggesting the knowledge that the learner should develop through engaging with the problem. Brousseau explains that part of teaching is to provide problems that *seem* too difficult for the learner. If the teacher tells the learner how to solve the problems and provides the method or procedure, the learners cannot learn it for themselves. Learners have to struggle through solving the problem to obtain the knowledge contained in it.

Schoenfeld *et al.* (2016) set out that teachers need to "position students as sense makers" (p. 2). The learners should be wrestling with ideas and solutions and they should be doing most of the mathematical work in the classroom. If the teacher provides hints, these hints should support productive struggle and provide learners with time to think things through (Kartal, Popovic, Morrissey & Holifiled, 2017). For Pauli (1960), an "effective teacher anticipates the kind of confusion the learner is capable of overcoming and plans his lesson accordingly" (p. 81). Furthermore, the teacher needs to create a milieu in the classroom in which learners *expect* to work through struggle.

Learner orientations for productive struggle

The link between learner beliefs and mathematics learning has been investigated (McDonough & Sullivan, 2014; Op'tEynde, De Corte & Verschaffel, 2002). Challenging tasks allow learners to develop empowering dispositions towards mathematics (Muraskwa, 2018). For learners to be engaged with productive struggle, they need to feel comfortable when sharing incomplete or incorrect work (Schoenfeld *et al.* 2016) and not to push the teacher for the specific steps to follow (Stein & Smith, 1998). They also need to provide explanations of what they have done when asking for help and to be specific about the assistance they require. They need to believe that learning mathematics means being willing to test and re-test ideas. Generally, learners' orientations may compromise their appreciation of productive struggle, for example, they believe that they should solve mathematical problems within five minutes by following the teacher's example (Schoenfeld, 1992).

Conclusion

Productive struggle does not mean frustration or utter confusion, but rather that learners are engaged, curious and motivated to pursue the learning that a challenging, well-chosen task holds. The most challenging aspect of design for productive struggle in mathematics classrooms may not be task selection, but rather shifting teacher and learner orientations towards learning *through* struggle. Classroom norms and the distribution of the mathematical work in classrooms must enable productive struggle. Demanding tasks should be accompanied by "room and support for growth" (Schoenfeld *et al.,* 2016, p. 5) so that learners build on their current knowledge without always relying on recipe-like solutions.

Confusion should be seen as "the catalytic agent that generates the problem-solving process" and the thinking required to solve the problem as "the solvent" (Pauli, 1960, p. 82). Productive struggle leads to deeper learning (Kartal *et al.,* 2017; Kapur, 2016) and more powerful sense-making systems (Lesh *et al.,* 2013, p.57), even if it makes success difficult in the short term (Kapur, 2016).

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Equivalence and substitution: tools for teaching meaningful mathematics

SU LIANG

In issue **40**(2) Melhuish and Czocher (2020) raise a very interesting discussion about the essential role of recognizing mathematical objects' sameness in mathematical teaching and learning. One example they give of a situation where ideas of sameness are important is recognizing 'expression sameness' in solving algebraic equations. Here I would like to consider sameness in a related context, equivalence and substitution across mathematical topics and grade levels. I have observed that the preservice teachers that I work with do not recognize the sameness and variation of mathematical expressions and ideas. I suggest some ways to address this.

Equivalence and substitution are cases of what Tall (2011) calls a 'crystalline concept'. A crystalline mathematical concept can be expressed in different forms by an equivalence of relationships in different contexts and at different learning stages. Tall emphasizes the importance of having mathematics learners "seek a broader understanding of the crystalline structures of mathematics" (p. 8). For students to seek such an understanding, they must be presented with suitable tasks, and to develop such task teachers must also have an understanding of the crystalline structures of mathematics. As teacher educators, we must equip our prospective teachers with crystalline structures which help them gain a broad vision of connected mathematics.

Equivalence and substitution across grade levels

Equivalence and substitution are often used to solve mathematical problems from primary level to college level. To name a few examples:

1. In arithmetic, we use equivalent fractions to add or subtract fractions without same denominators:

   $$\frac{2}{3}+\frac{1}{2}+\frac{3}{4}=\frac{4}{12}+\frac{6}{12}+\frac{9}{12}=\frac{19}{12}=1\frac{7}{12}\,.$$

   In this operation,

   $$\frac{2}{3}=\frac{4}{12},\frac{1}{2}=\frac{6}{12},\frac{3}{4}=\frac{9}{12},$$

   therefore we can substitute $^{4}/_{12}$ for $^{2}/_{3}$, substitute $^{6}/_{12}$ for $^{1}/_{2}$, substitute $^{9}/_{12}$ for $^{3}/_{4}$, because they are equivalent fractions. We write the answer as $1^{7}/_{12}$ which is equivalent to improper fraction $^{19}/_{12}$. Arithmetic problems are often solved using the ideas of equivalence and substitution.

2. In algebra, substitution is an important technique to solve systems of equations. Substitution can also be seen in the idea of using variables to represent numbers. Equivalent forms are very useful for solving various algebraic equations. For example,

   $$a^2-b^2=(a-b)(a+b);$$
   $$(a+b)^2=a^2+2ab+b^2;$$
   $$(a-b)^2=a^2-2ab+b^2;$$
   $$x^2+bx+c=a(x-\frac{b}{2a})^2+\frac{4ac-b^2}{4a};$$
   $$(x-a)^2+(y-b)^2-r^2=$$
   $$x^2-2ax+y^2-2by+a^2+b^2-r^2=0.$$

   Equivalent forms are powerful tools that can help transform a complicated mathematical equation into a form that is simpler to solve.

3. In geometry, if polygon A is congruent to polygon B and polygon B is congruent to polygon C, then polygon A is congruent to polygon C. Many geometric proofs use such ideas of equivalence and substitution.

4. In trigonometry, many trigonometric identities can be proved using the idea of equivalence and substitution.

5. In calculus, substitution is a very powerful method for finding integrals. Many theorems are proved using the idea of equivalence and substitution.

Wasserman and Weber (2017) proposed that in teacher education we should help teachers develop "not just mathematical knowledge but also particular pedagogical aptitudes" (p. 18) by helping them understand the connections between advanced mathematics they learn and the secondary teaching contents they will teach. Identifying situations of equivalence and substitution is such an aptitude. Developing it will help the preservice teachers to connect previously learned knowledge into a coherent whole and to gain a bigger picture of mathematics. We can use equivalence and substitution as a thread to pull different levels of mathematical concepts together (see Figure 1).

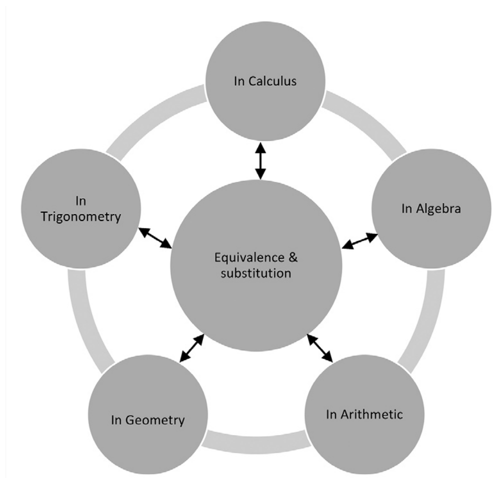

Figure 1. Equivalence and substitution pulling together mathematical concepts.

Understanding the idea of equivalence and substitution

Here is an open-ended homework question from my capstone courses in recent semesters:

> Please reflect on all mathematics you have learned so far. Create a map that demonstrates the big idea of Equivalence. Be creative, I want to see some great maps from you.

Most of the preservice teachers responded with some equivalent numbers such as:

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = 0.3333\ldots\ldots = 33.33\ldots\ldots\%$$

or some equivalent simple algebraic expressions:

$$x^3 = \frac{x^5}{x^2} = 2x^3 - x^3 = \sqrt{x^6} = \sqrt[3]{x^9}$$

These responses either articulated the relationships among mathematical concepts of fractions, decimals, and percentage or listed algebraic operations using different algebraic rules. However, I had expected that the preservice teachers, who had completed Calculus, Geometry, Linear Algebra, and Abstract Algebra, would have had a better vision of equivalence. There was one response that was closer to what I expected (Figure 2). Starting from the the number 1, the student listed various equivalent representations including numeric operations, position on a number line, fractions, algebraic expressions, trigonometric expressions, and expression from Calculus.

Although there are a few careless errors, the response in Figure 2 demonstrates how powerful the idea of equivalence and substitution is in connecting mathematical ideas and concepts across subject areas and grade levels, even starting from a value as simple as 1. Being able to recognize such equivalent relationships between different forms across topics and grade levels is a sign of high level mathematical thinking. I noticed that the student who made this response performed much better than the others in my class.

Equivalence and substitution in class activities

Why did so few of my students make the connections shown in Figure 2? What can be done to provide opportunities for preservice teachers to to develop broader and deeper understanding of equivalence and substitution? I offer a sample class activity to illustrate my attempts to do so. The activity has four parts:

Part I: Using two methods to express the repeating decimals with fractions:

0.6666…; 0.7777…; 0.8888…; 0.161616…; 0.353535…

Part II: Can you predict the fraction for

0.5555…; 0.565656…; and 0.123123123…

without calculation? Explain your reason.

Part III: Reflect on the two methods we applied, what mathematics ideas/concepts are needed to apply these two methods?

Part IV: Homework: Prove that

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r}$$

(a is a constant not equal to 0 and $|r| < 1$)

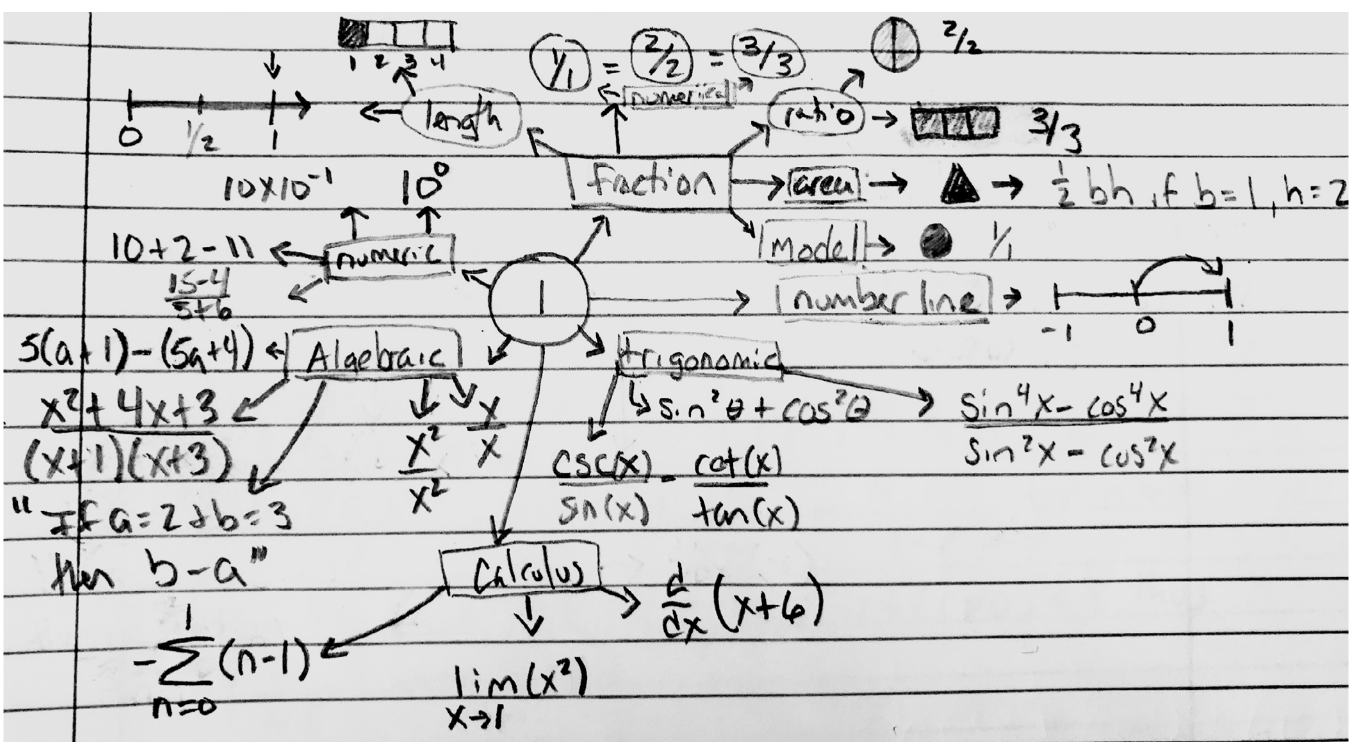

Figure 2. A rich idea of equivalence.

Part I requires the preservice teachers to use two methods. The first method is the usual one they might have seen in school:

$$\text{Let} \quad x = 0.6666\ldots \qquad ①$$

$$\text{Multiply both sides of ① by 10: } 10x = 6.6666\ldots \qquad ②$$

$$\text{Subtract ① from ②:} \quad 9x = 6 \qquad ③$$

$$\text{Divide 9 in both sides of ③:} \quad x = 0.6666\ldots = \tfrac{6}{9} = \tfrac{2}{3}$$

This method involves using the properties of equations to generate equivalent equations. It is also a context to discuss how equivalence differs from equality. When ① is subtracted from ② the result is a new equivalent equation, unlike when a number is subtracted from an equal number, resulting in 0.

The second method involves recognizing the equivalence of a repeating decimal and a geometric series:

$$0.6 + 0.06 + 0.006 + 0.0006 + \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \cdots$$

This makes it possible to apply techniques for finding the closed form of a geometric series that my students have studied in Calculus II.

Part II engages the preservice teachers in observing the patterns of the results for the decimals with one repeating digit, two repeating digits, and three repeating digits and then generalizing the conclusions. Doing so they experience that the same solution method can be applied to different numbers, and that when the number of digits changes there is an equivalent method they can apply.

Part III pushes the preservice teachers to think further about:

1. the connections between decimals and fractions, between repeating decimals and rational numbers, between expansion of repeating decimals and the concept of place value, and between algebraic methods;
2. how the ideas of equivalence and substitution are used across different mathematical contexts;
3 how repeating decimals are related to infinite series or more generally how different forms of equivalent mathematical expressions can make different techniques available.

Part IV requires the preservice teachers to deduce the formula for the closed form of a geometric series. Though they might remember the formula (which they use in Part II) most of my students have forgotten how it is proved. Revisiting this proof can help them see how mathematics is developed and connected across grade levels and subject areas, and specifically in this case, how repeated decimals in school mathematics are related to geometric series in university level mathematics.

This class activity extends the dimension of expected learning for the preservice teachers either horizontally (across different mathematical concepts, ideas and methods) or vertically (across school levels). The process of solving the problems in the activity and reflecting on their solutions provides the preservice teachers opportunities to review key mathematical contents, apply the ideas of equivalence and substitution, use different mathematical methods, connect mathematical concepts across different grade levels and gain a vision of interconnected mathematical ideas.

Conclusion

Exploring mathematical sameness can provide preservice teachers with a lens for better understanding while teaching (Melhuish & Czocher, 2020). Mathematical equivalence, specifically, can be used as a thread to connect mathematical concepts and topics. Much remains to be done to analyze the nature of equivalence and substitution in different contexts, and in developing suitable tasks to provide students (including preservice teachers) with opportunities to enrich their understandings of these ideas.

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Linear algebra students' conceptions: sets that are closed under operations

CARMIT BENBENISHTY

My work as a mathematics lecturer and my research studying undergraduate students' thinking patterns led me to the realization that many undergraduate linear algebra students tend to, incorrectly, consider many sets to be closed under a given operation. This communication aims to shed light on this interesting conceptual pattern, to present justifications given by linear algebra students that illustrate this conception, and to offer possible sources for it.

Students' conception about sets that are closed under a certain operation

Closure is an important concept in linear algebra. Closure of a set S under an operation $*$ means that, given two elements a and b in S and an element c which is obtained from a and b by the operation $*$, c must also be in S. While learning linear algebra, students encounter many sets that are closed under a given operation. For example, they learn that the product and the sum of any two square matrices of the same order is also a square matrix of the same order; in other words, the set of square matrices of order n is closed under addition and multiplication. They also learn that the set of invertible matrices of order n is closed under multiplication, meaning that the product of two invertible matrices of a given order is an invertible matrix of the same order. Thinking about closure can be a useful way to approach problems in linear algebra.

However, not all sets are closed under all operations, and yet many students wrongly assume that they are.

In my work I have encountered many examples of students incorrectly assuming closure, claiming, for example, the following:

- The sum of two irrational numbers is inevitably an irrational number. After seeing the counter-example $\sqrt{2} + -\sqrt{2} = 0$, students adjust their assumption only slightly, assuming that the set of irrationals is closed under addition *unless* the two irrational numbers are additive inverses.
- The sum of any two singular matrices is singular.
- The sum of two invertible matrices A and B is invertible, though some students exclude the case where $A = -B$.
- If the only solution of each of the homogeneous systems $Ax = 0$ and $Bx = 0$ is the zero solution, then the only solution of the homogeneous system $(A + B)x = 0$ is the zero solution.
- If each of the homogeneous systems $Ax = 0$ and $Bx = 0$ has infinitely many solutions, then so does the homogeneous system $(A + B)x = 0$.
- Given two square matrices A and B satisfying $det(A)$, $det(B) > 0$, then necessarily $det(A + B) > 0$.
- The sum and the product of any two real diagonalizable matrices of the same order is also diagonalizable. A common justification is that the sum and the product of two diagonal matrices are also diagonal matrices. Another common justification is that if $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$, then $P^{-1}(A+B)P = D_1 + D_2$ and $P^{-1}(A \cdot B)P = D_1 \cdot D_2$.

Similar inferences are made regarding the concept of eigenvector. Many students who are given the question "Are there two square matrices A and B and a vector v such that v is an eigenvector of A and B but v is not an eigenvector of $A + B$?" respond that if v is an eigenvector of A and B, then v is necessarily an eigenvector of $A + B$. A repeated justification is that the equations $Av = cv$ and $Bv = cv$ imply $(A + B)v = cv$. In addition, many students mistakenly think that, given a square matrix A, the sum of two non-eigenvectors cannot be an eigenvector of A.

Possible sources for this conception

One possible source for this conception is the insufficient comprehension of the various concepts included in a typical linear algebra course. There is a possibility that students' familiarity with each of these concepts is not deep enough, and therefore they do not have enough tools to cope with certain questions about them. This situation may be a consequence of the way a linear algebra course is taught, if many concepts are covered too quickly without first giving them an intuitive basis (Harel, 1989), or if they are introduced without significant examples or applications (Carlson, 1993).

For instance, one of the fundamental mistakes of students who argue that the set of real diagonalizable matrices of order n is closed under addition and multiplication is their disregard of the fact that two diagonalizable matrices are not necessarily simultaneously diagonalizable. This may be due to insufficient experience with the concept of diagonalization of matrices (Carlson, 1993).

Another possibility is related to the two systems of thinking presented by Kahneman (2011). According to Kahneman, there are two different ways in which our brains form thoughts: "System 1 operates automatically and quickly, with little or no effort and no sense of voluntary control [while] System 2 allocates attention to the effortful mental activities that demand it, including complex computations" (pp. 20–21). Kahneman outlines the relationship between these two systems:

> I describe System 1 as effortlessly originating impressions and feelings that are main sources of the explicit beliefs and deliberate choices of System 2. The automatic operations of System 1 generates surprisingly complex patterns of ideas, but only the slower System 2 can construct thoughts in an orderly series of steps. [In some circumstances] System 2 takes over, overruling the freewheeling impulses and associations of System 1. (p. 21)
>
> When System 1 runs into difficulty, it calls on System 2 to support more detailed and specific processing that may solve the problem of the moment. System 2 is mobilized when a question arises for which System 1 does not offer an answer [...] System 2 is mobilized to increased effort when it detects an error about to be made. (pp. 24–25)

Although System 1 is very good at what it does, it tends to have biases and systematic errors in specific circumstances. It sometimes responds to easier questions than those it was asked and it has little understanding in logic.

A key difference between students who falsely assume that sets are closed, and students who do not, may be a different division of labor between System 1 and System 2. In order to justify correctly whether or not a set is closed under a given operation, it is necessary to keep in memory several concepts in that need to be combined, according to specific properties and rules. Kahneman argues that, in this case, cognitive effort is required, and System 2 is the only one that can be used. In order to avoid the wrong inferences that System 1 may make, attention to System 2 is required. It is possible that System 1 is the one responsible for the wrong impression that a set is closed under a certain operation and that a major disparity between students whose justifications are incorrect and those whose justifications are correct, stems from attention to System 2. Perhaps, students whose justifications are correct attend System 2, and, as a result, System 2 operates and overcomes the wrong conclusions of System 1, whereas among students whose justifications are incorrect, System 2 does not overcome System 1 and it does not detect the wrong assumptions made by System 1.

A third possible source for this conception can be found in the theory of the three conceptual worlds outlined by David Tall (2004, 2013). The three conceptual worlds of Tall describe a hierarchy of three forms of thinking. The *embodied* world is where perception is developed through mental embodiment of mathematical objects using our daily experience. The *symbolic* world is where we operate on the symbolic level, and eventually we comprehend those operations at such a level that they do not require much conscious effort. Finally, the *formal* world is where we use formal reasoning and axiomatic thinking. According to Tall, ideally students first develop their understanding through embodiment of the underlying mathematical objects, they then continue and learn how to symbolize these objects and how to operate on them in the symbolic world and finally, they learn to formally reason about them.

Beltrán-Meneu, Murillo-Arcila and Albarracín (2017) use Tall's framework to analyze and classify students' answers to the following question:

> Consider the following matrix
>
> $$\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$$
>
> and consider the vectors $v_1 = (2, 0)$, $v_2 = (2, 1)$, $v_3 = (2, 2)$. Are the vectors $v_1 + v_2$ and $v_2 + v_3$ eigenvectors of A?

The researchers classified students' responses as either symbolic or formal. Students who looked for such that $A(v_1 + v_2) = (v_1 + v_2)$ and $A(v_2 + v_3) = (v_2 + v_3)$ were identified as using "the formal definition of an eigenvector, but they proceeded in a symbolic way to give the answer". Responses were categorized as formal for students who "reasoned that the sum of eigenvectors is an eigenvector if and only if all vectors belong to the same subspace" (p. 129).

The responses mentioned above, of students who argue that the sum and the product of any two real diagonalizable matrices of order n is also diagonalizable and who justify this by referring to the formal definition of a diagonalizable matrix, could be identified as formal. However, it is important to notice that these students omit a critical factor in the definition. While trying to justify their claim, they use the formal definition, but disregard the case where there is no common invertible matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices. Other students only refer to the case in which the matrices A and B are diagonal, completely disregarding the case in which A and B are not diagonal. One may classify this type of response as symbolic.

In conclusion

The concept of closure is clearly difficult for linear algebra students, but linear algebra is not the first context in which closure is a relevant concept. Already in elementary schools, children learn that it is not always possible to find a natural number quotient when one divides a natural number by another natural number. They could learn (without using the terminology) that the set of rational numbers is closed under division though the natural numbers are not. Similarly, middle school students learn about integers and that the set of integers is closed under subtraction even though the natural numbers are not. Perhaps earlier discussion of closure will help students to see that closure of a set cannot be assumed.

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From the archives

The following is an edited excerpt from *The Interactive Monitoring of Children's Learning of Mathematics* by David Clarke, in issue **7**(1), pp. 2–6.

Here are some questions we never ask our mathematics students:

- What was the best thing to happen in Maths?
- What is the biggest worry affecting your work in Maths?
- What is the most important thing you have learnt in Maths?
- How do you feel in Maths classes?
- How could we improve Maths classes?

Yet it is the answers to questions like these which could more usefully guide the planning of mathematics instruction than many of the content-based questions we do ask.

During 1984 about 700 children in 36 first-year mathematics classes in 15 Victorian secondary schools were regularly given the opportunity, about once every two weeks, to give confidential written answers to questions like the ones above.

Their replies were funny and moving, trivial and profound. Their teachers were often placed in a dilemma. How do you respond to a child who writes,

> I don't know what is wrong but I think it is going in one ear and out the other. How can I improve when I don't understand? I want to improve and pass year 7 so much. Can you help me?

[…]

The IMPACT procedure required pupils to give confidential (but not anonymous), written responses, fortnightly, to two alternate sets of four simple questions [...]. In doing this the children had to reflect and report on their anxieties and successes in secondary mathematics, on the difficulties they experienced, and on the quality of the instruction they received.

[…]

The children's responses graphically illustrate many of the issues currently occupying the attention of mathematics educators. The emergence of these issues in the writings of the pupils in contemporary mathematics classes endows each with an immediacy often missing in the cautious, considered words of educational research, and reminds us of our obligations to those who are the subject, the justification and ultimately the beneficiaries of our efforts.

All of the quotations which follow were taken from children's actual answers to the two sets of four questions […]. Each quote was included because it was representative of a class of similar responses, exemplified a particular student perspective, or raised a significant issue. […] In each case the sex of the child is indicated, together with the month during which the statement was made. It should be noted that the school year in Australia runs from February to December.

> I like being in the middle room because I don't like being in the brainy or the dumb room. Girl (February)
>
> The teacher works very slowly and I think that's better I can understand better. Girl (February)
>
> The things that I liked best were firstly factors and secondly divisions. I liked them because I understood them. Girl (February)
>
> [The thing I would most like help with is] Fractions but the teacher thinks I know them. Boy (June)
>
> Main thing in Maths I have problems with dividing. Boy (March)
>
> Fractions and Long Division. Same Boy (September)
>
> Long Division and Fractions. Same Boy (November)
>
> [My biggest worry is] Keeping up with the rest of the class. Girl (February and every month thereafter)
>
> In tests I get a bit nervous and my mind goes fuzzy. Boy (May)
>
> I am not sure but I always seem to do something wrong in my graphs but cannot work out what I am doing wrong. Girl (August)
>
> [My biggest worry is] Passing 2nd term and getting a good report so mum and dad will be proud of me. Girl (August)
>
> [My biggest worry is] My dad has been away for six weeks now in a war exercise overseas in Europe. Girl (September)
>
> [The most important thing I have learnt in maths in the last two weeks is] I'm stupid in class. Girl (November)
>
> I don't really think I've learnt anything very important to me. Because I don't like maths. SORRY. Girl (March)
>
> [How do you feel in Maths classes at the moment?] Confused. Bored. Worried, Rushed. DUM. In other words I'm stuped. Girl (July)
>
> Interested. Right now I feel teribile, awful, rotten. It's got nothing to do with Maths but it's in the way. Girl (September)
>
> Relaxed. Bored. I feel relaxed because I'm bored. Boy (March)
>
> Bored. Angry. (If you're wondering why I'm angry it's because I don't like being bored). Girl (March)
>
> [How could we improve maths classes?] Have less work and more learning. Boy (September)

Some observations […]

- While many children gave serious, thoughtful responses to the IMPACT items, some students were unwilling or unable to make useful responses. Some teachers put forward the conjecture that students with limited language skills had difficulty articulating their concerns and found the need for written responses a burden.
- Detailed examination of students' actual responses supported teachers' observations that girls were more likely than boys to make useful responses, however a higher proportion of boys than girls reported finding the IMPACT procedure personally useful. [...]
- Several instances were reported in which teacher action arising from information obtained through the IMPACT procedure led to positive changes in students' attitudes and achievement. Lack of teacher response, on the other hand, was the single complaint voiced by those students dissatisfied with the IMPACT program. [...]
- While the quality and character of the children's responses was extremely varied, many of the participating students made responses which were informative and showed real insight.

[…]

There were clearly instances where teacher action in response to student requests or suggestions significantly altered the form of instruction. Students in those classes were confronted with the need for a reinterpretation of their role, and the idea of students as "active participants" (rather than passive recipients) took on an added meaning.

Editor's Note

2020 was a year of many losses. Here I'd like to mark the passing of two outstanding Australian scholars, Judy Mousley and David Clarke. Judy's passion for supporting disadvantaged students resonates with many of the articles in this issue. In her research and her teaching she advocated for the inclusion of all students and her work with indigenous Australian students is well known internationally. David's article, excerpted above, shows his lifelong interest in listening to the voices of learners. He is perhaps best known for the Learner's Perspective Study, which focussed on the voices of students in 16 countries. Both were important leaders in the mathematics education community. They will be missed.